数理解析研究所講究録
730 巻 1990 年 183-198

BEHAVIOR OF FREE BOUNDARIES OF STEFAN PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

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0. Introduction

In the paper [1], we established a local existence result and a comparison result for the following two-phase Stefan problem in one-dimensional space: Find a function $u = u(t,x)$ on $Q(T) = (0,T) \times (0,1)$, $0 < T < \infty$, and a curve $x = l(t)$, $0 < l < 1$, on $[0,T]$ such that

$$\text{(0.1)} \qquad \rho(u)_t - a(u_x)_x + h(t,x) = \begin{cases} f_0(t,x) & \text{in } Q_l^+(T), \\ f_1(t,x) & \text{in } Q_l^-(T), \end{cases}$$

$$Q_l^+(T) = \{(t,x);\ 0 < t < T,\ 0 < x < l(t)\},$$

$$Q_l^-(T) = \{(t,x);\ 0 < t < T,\ l(t) < x < 1\},$$

$$\text{(0.2)} \qquad h(t,x) \in g(t,x,u(t,x)) \qquad \text{for a.e. } (t,x) \in Q(T),$$

$$\text{(0.3)} \qquad \begin{cases} u(t,l(t)) = 0 & \text{for any } t \in [0,T], \\ l'(t) = -a(u_x(t,l(t)-)) + a(u_x(t,l(t)+)) & \text{for a.e. } t \in [0,T], \\ l(0) = l_0, & \end{cases}$$

$$\text{(0.4)} \qquad \begin{cases} a(u_x(t,0+)) \in \partial b_0^t(u(t,0)) & \text{for a.e. } t \in [0,T], \\ -a(u_x(t,1-)) \in \partial b_1^t(u(t,1)) & \text{for a.e. } t \in [0,T], \end{cases}$$

$$\text{(0.5)} \qquad u(0,x) = u_0(x) \qquad \text{for any } x \in [0,1],$$

where ρ: $R \to R$ and a: $R \to R$ are continuous increasing functions; $g(t,x,\cdot)$ is a given set-valued mapping in R for a.e. $(t,x) \in Q(T)$; $f_i (i = 0, 1)$ is a function on $Q(T)$; l_0 is a number with $0 < l_0 < 1$ and u_0 is a function on $[0,1]$; $b_i^t (i = 0, 1)$ is a proper l.s.c. convex function on R for each $t \in [0,T]$ and ∂b_i^t denotes its subdifferential in R.

It should be noticed that (0.4) represents various linear or nonlinear boundary conditions, such as Dirichlet, Neumann and Signorini type of boundary conditions (cf. [1]).

In order to investigate the behavior of the free boundary $x = l(t)$, we denote by $[0,T^*)$ the maximal interval of existence of solution to the

Stefan problem (0.1) ~ (0.5). In Kenmochi [2] or Knabner [3], it was proved for the Stefan problem with $g(t,x,r) \equiv 0$ that one and only one of the following cases always happens:

(A) $T^* = \infty$;

(B) $T^* < \infty$, and $l(t) \to 0$ as $t \uparrow T^*$;

(C) $T^* < \infty$, and $l(t) \to 1$ as $t \uparrow T^*$.

But, in our problem, the following case (D) might occur:

(D) $T^* < \infty$, $0 < \inf\{l(t);\ 0 \le t < T^*\} \le \sup\{l(t);\ 0 \le t < T^*\} < 1$ and

$$\limsup_{t \uparrow T^*} |u(t)|_{L^\infty(0,1)} = \infty,$$

because the class of functions $g(t,x,r)$ includes any locally Lipschitz continuous function on R and in general equation (0.1) has a solution which blows up at finite time. Therefore it is quite natural to estimate the function u by means of the solution v to the following initial-boundary value problem $(IBP)_0$ (resp. $(IBP)_1$) which is formulated by

$$\rho(v)_t - a(v_x)_x + h = f_0 \quad (\text{resp. } f_1) \qquad \text{in } Q(T),$$

$$h(t,x) \in g(t,x,v(t,x)) \qquad \text{for a.e. } (t,x) \in Q(T),$$

$$a(v_x(t,0+)) \in \partial b_0^t(v(t,0)) \qquad \text{for a.e. } t \in [0,T],$$

$$(\text{resp. } v(t,0) = 0 \qquad \text{for } 0 \le t \le T)$$

$$v(t,1) = 0 \qquad \text{for } 0 \le t \le T,$$

$$(\text{resp. } -a(v_x(t,1-)) \in \partial b_1^t(v(t,1)) \qquad \text{for a.e. } t \in T)$$

$$v(0,x) = u_0^+(x) \quad (\text{resp. } -u_0^-(x)) \qquad \text{for } 0 \le x \le 1.$$

As is well known, for $i = 0, 1$ the solution of $(IBP)_i$ in general blows up at some finite time. For $i = 0, 1$, let $[0, T_0^{(i)})$ be the maximal interval of existence of solution to $(IBP)_i$, and put $T_0 = \min\{T_0^{(0)}, T_0^{(1)}\}$. Then our main result is stated as follows: One and only one of the following cases (a), (b), (c) always happens:

(a) $T^* \ge T_0$;

(b) $T^* < T_0$ and $l(t) \to 0$ as $t \uparrow T^*$;

(c) $T^* < T_0$ and $l(t) \to 1$ as $t \uparrow T^*$.

1. Statement of results

In general, for a Banach space V, we denote by $|\cdot|_V$ the norm in V.

Throughout this paper, for the sake of simplicity of notations we put

$$H = L^2(0,1) \text{ and } X = W^{1,p}(0,1) \ (\subset C([0,1])),\ 2 \le p < \infty.$$

Furthermore, for two proper l.s.c. convex functions $b_1(\cdot)$ and $b_2(\cdot)$ on R we mean by "$b_1 \le^* b_2$ on R" that

$$(1.1)\quad \begin{cases} (r_1' - r_2')(r_1 - r_2)^+ \ge 0 \text{ for any } r_i \in D(\partial b_i),\ r_i' \in \partial b_i(r_i), \\ i = 1,\ 2, \end{cases}$$

and for two set-valued functions $g_1(\cdot)$ and $g_2(\cdot)$ on R we mean by "$g_1 \le^{**} g_2$" that for each $M > 0$ there is a positive constant C_M^* such that

$$(1.2)\quad \begin{cases} (r_1' - r_2')(r_1 - r_2)^+ + C_M^*|(r_1 - r_2)^+|^2 \ge 0 \text{ for any } r_i \text{ with} \\ |r_i| \le M,\ r_i' \in g_i(r_i),\ i = 1,\ 2. \end{cases}$$

We denote by $SP = SP(\rho;\ a;\ b_0^t,\ b_1^t;\ g;\ f_0,\ f_1;\ u_0,\ l_0)$ on $[0,T]$, $0 < T < \infty$, the problem (0.1) ~ (0.5), and say that $\{u,\ l\}$ is a solution of SP on $[0,T]$, if

$$\begin{cases} u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X) \ (\subset C([0,T] \times [0,1])), \\ l \in W^{1,2}(0,T),\ 0 < l < 1 \text{ on } [0,T], \end{cases}$$

and (0.1) ~ (0.5) are satisfied. Also, we say that for $0 < T' \le \infty$, $\{u,\ l\}$ is a solution of SP on $[0,T')$, if it is a solution of SP on $[0,T]$ for any T with $0 < T < T'$ in the above sence.

We begin with the precise assumptions (a1) ~ (a6) on ρ, a, g, f_i $(i = 0,\ 1)$, $b_i^t(i = 0,\ 1)$, u_0 and l_0 under which Stefan problem (0.1) ~ (0.5) is discussed.

(a1) $\rho\colon R \to R$ is a bi-Lipschitz continuous and increasing function with $\rho(0) = 0$; denote by C_ρ a Lipschitz constant of ρ and ρ^{-1}.

(a2) $a\colon R \to R$ is a continuous function such that

$a_0|r|^p \le a(r)r \le a_1|r|^p$ for any $r \in R$,

$a_0(r - r')^{p-1} \le a(r) - a(r')$ for any $r,\ r' \in R$ with $r \ge r'$,

where a_0 and a_1 are positive constants.

(a3) We assume that for a.e. $(t,x) \in R_+ \times (0,1)$ the mapping $r \to g(t,x,r)$ is a set-valued mapping in R such that $g(t,x,r)$ is a non-empty closed interval in R for any $r \in R$, $0 \in g(t,x,0)$ and $g(t,x,r)$ is u.s.c. with respect to $r \in R$. Moreover suppose that for each $M > 0$, g has the following properties (i) ~ (iii):

(i) $r \to g(t,x,r) + C_M r$ is monotone in r with $|r| \le M$ for a positive constant C_M depending only on M, that is,

$(r_1' + C_M r_1 - r_2' - C_M r_2)(r_1 - r_2) \ge 0$ if $|r_i| \le M$ and $r_i' \in g(t,x,r_i)$, $i = 1,\ 2$.

(ii) $|r'| \le g_{0,M}(t,x)$ for $r' \in g(t,x,r)$, r with $|r| \le M$ and a.e. $(t,x) \in R_+ \times (0,1)$, where $g_{0,M}$ is a non-negative function in $L^2_{loc}([0,\infty);H)$;

(iii) for any λ with $0 < \lambda < 1/C_M$ and r with $|r| \le M$, $[1 + \lambda g(t,x,\cdot)]^{-1}r$ is measurable in $(t,x) \in (0,\infty) \times (0,1)$,

(a4) $f_0, f_1 \in L^2_{loc}([0,\infty);H) \cap L^1_{loc}([0,\infty);L^\infty(0,1))$,
$f_0 \ge 0$, $f_1 \le 0$ a.e. on $(0,\infty) \times (0,1)$,

(a5) For $i = 0, 1$ and each $t \in [0,\infty)$ b^t_i is a proper l.s.c. convex function on R which satisfies the following condition (*) for given functions $\alpha_0 \in W^{1,2}_{loc}([0,\infty))$, $\alpha_1 \in W^{1,1}_{loc}([0,\infty))$:

(*) For any $0 \le s \le t < \infty$ and $r \in D(b^s_i) \equiv \{r \in R;\ b^s_i(r) < \infty\}$ there exists $r' \in D(b^t_i)$ such that

$$|r' - r| \le |\alpha_0(t) - \alpha_0(s)|(1 + |r| + |b^s_i(r)|^{1/p}),$$
$$b^t_i(r') - b^s_i(r) \le |\alpha_1(t) - \alpha_1(s)|(1 + |r|^p + |b^s_i(r)|).$$

Furthermore, we suppose that
$\partial b^t_0(r) \subset (-\infty, 0]$ for any $r < 0$ and $t \in [0,\infty)$,
and
$\partial b^t_1(r) \subset [0,\infty)$ for any $r > 0$ and $t \in [0,\infty)$.

(a6) $0 < l_0 < 1$, $u_0 \in X$, $u_0(i) \in D(b^0_i)$, $i = 0, 1$, and $0 \le u_0$ on $[0, l_0]$, $u_0 \le 0$ on $[l_0, 1]$.

__REMARK 1.1__. *(cf. [1]) Under the assumptions (a1) ~ (a6), there is a positive constant T' such that $(IBP)_0$ (resp. $(IBP)_1$) has a unique solution v on $[0,T']$ and $v \in W^{1,2}(0,T';H) \cap L^\infty(0,T';X) \subset C([0,T'] \times [0,1])$.*

__THEOREM 1.1__. *Under the assumptions (a1) ~ (a6), denote by $[0,T^*)$, $0 < T^* \le \infty$, the maximal interval of existence of solution $\{u, l\}$ to SP. Then, one and only one of the following cases (a), (b), (c) always happens:*

(a) $T^* \ge T_0$;

(b) $T^* < T_0$ *and* $l(t) \to 0$ *as* $t \uparrow T^*$;

(c) $T^* < T_0$ *and* $l(t) \to 1$ *as* $t \uparrow T^*$,

where $T_0 = \min\{T_0^{(0)}, T_0^{(1)}\}$ *and for* $i = 0, 1$, $[0, T_0^{(i)})$ *is the maximal interval of existence of solution to* $(IBP)_i$.

2. __Known results__

We begin with a local existence result for SP.

__THEOREM 2.1__. *(cf. [1]) Suppose that (a1) ~ (a6) hold. Moreover, for*

$T > 0$, we assume that

$$|f_0|_{L^2(0,T;H)},\ |f_1|_{L^2(0,T;H)},\ |\alpha_0|_{W^{1,2}(0,T)},\ |\alpha_1|_{W^{1,1}(0,T)} \le K_1,$$

$$|u_0|_{L^\infty(0,1)} \le M - 1,$$

$$|g_{0,M}|_{L^2(0,T;H)} \le K_2, \text{ and } \delta_0 \le l_0 \le 1 - \delta_0,$$

where K_1, K_2, M and δ_0 are positive constants. Then there exists a positive number T_1 with $T_1 \le T$ depending only on K_1, K_2, M and δ_0 such that SP has a unique solution $\{u, l\}$ on $[0,T_1]$ and $\delta_0/2 \le l(t) \le 1 - \delta_0/2$ on $[0,T_1]$.

Next we recall a comparison result for solutions to SP.

THEOREM 2.2. *(cf. [4]) Let ρ and a be functions satisfying (a1) and (a2), respectively, and consider the Stefan problem $SP = SP(\rho;\ a;\ b_0^t,\ b_1^t;\ g;\ f_0,\ f_1;\ u_0,\ l_0)$ and $\overline{SP} = SP(\rho;\ a;\ \bar b_0^t,\ \bar b_1^t;\ \bar g;\ \bar f_0,\ \bar f_1;\ \bar u_0,\ \bar l_0)$, where the set of data $\{b_0^t,\ b_1^t,\ g,\ f_0,\ f_1,\ u_0,\ l_0\}$ as well as $\{\bar b_0^t,\ \bar b_1^t,\ \bar g,\ \bar f_0,\ \bar f_1,\ \bar u_0,\ \bar l_0\}$ satisfies (a3) ~ (a6). Further suppose that*

$$b_i^t \le^* \bar b_i^t \text{ on } R \text{ for } i = 0,\ 1 \text{ and each } t \ge 0,$$

$$g(t,x,\cdot) \le^{**} \bar g(t,x,\cdot) \quad \text{for a.e. } (t,x) \in (0,\infty) \times (0,1),$$

and

$$f_0 \le \bar f_0,\ f_1 \le \bar f_1 \quad \text{a.e. on } (0,\infty) \times (0,1).$$

Let $\{u,\ l\}$ and $\{\bar u,\ \bar l\}$ be solutions of SP and $\overline{SP}$ on $[0,T']$, $0 < T' < \infty$, respectively. Then, we have for any $0 \le s \le t \le T'$

$$|[\rho(u(t)) - \rho(\bar u(t))]^+|_{L^1(0,1)} + [l(t) - \bar l(t)]^+$$

$$\le \{|[\rho(u(s)) - \rho(\bar u(s))]^+|_{L^1(0,1)} + [l(s) - \bar l(s)]^+\} \times \tag{2.1}$$

$$exp\{CC_\rho(t - s) + \textstyle\int_s^t[|f_0(\tau)|_{L^\infty(0,1)} + |f_1(\tau)|_{L^\infty(0,1)}]d\tau\},$$

where M is any constant with $|u| < M$ amd $|\bar u| < M$ on $[0,T'] \times [0,1]$ and $C = \max\ \{C_M^*,\ C_M,\ \bar C_M\}$ with constant C_M^* in condition (1.2) and constants C_M, $\bar C_M$ in condition (a3)-(i) corresponding to the data of SP, $\overline{SP}$, respectively.

3. Some lemmas

In this section we study the properties of solutions to the initial-boundary value problem $CP_0(l;\ b_0^t;\ g;\ f_0;\ u_0)$ formulated below:

$$\rho(u)_t - a(u_x)_x + h = f_0 \quad \text{in } Q_l^+(T),$$

$$h(t,x) \in g(t,x,u(t,x)) \text{ for a.e. } (t.x) \in Q_l^+(T),$$

$$a(u_x(t,0+)) \in \partial b_0^t(u(t,0)) \quad \text{for a.e. } t \in [0,T],$$

$$u(t,x) = 0 \qquad \text{for } (t,x) \in \overline{Q_l^-(T)},$$

$$u(0,x) = u_0(x) \qquad \text{for } 0 \le x \le 1,$$

where $x = l(t)$ is a given curve in $C([0,T])$ such that $0 < l(t) < 1$ and u_0 is a given datum in X.

<u>LEMMA 3.1</u>. *Assume (a1) ~ (a5) hold, and* $u_0 \in X$, $u_0(0) \in D(b_0^0)$ *and* $u_0(x) \ge 0$ *on* $[0, l(0)]$, $u_0(0) = 0$ *on* $[l(0), 1]$.
Let u be a solution of $CP_0(l;\ b_0^t;\ g;\ f_0;\ u_0)$ *in* $W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$. *Then*

$$u \ge 0 \quad \text{on } Q_l^+(T).$$

<u>LEMMA 3.2</u>. *Let ρ and a be functions satisfying (a1) and (a2), respectively, and consider the initial-boundary value problems* $CP_0 = CP_0(l;\ b_0^t;\ g;\ f_0;\ u_0)$ *and* $\overline{CP}_0 = CP_0(\bar{l};\ \bar{b}_0^t;\ \bar{g};\ \bar{f}_0;\ \bar{u}_0)$, *where the set of data* $\{l,\ b_0^t,\ g,\ f_0,\ u_0\}$ *as well as* $\{\bar{l},\ \bar{b}_0^t,\ \bar{g},\ \bar{f}_0,\ \bar{u}_0\}$ *satisfies the same assumptions as in Lemma 3.1. Further suppose that for* $0 < T < \infty$,

$$l \le \bar{l} \quad \text{on } [0,T],$$

$$b_0^t \le^* \bar{b}_0^t \text{ on } R \text{ for any } t \in [0,T],$$

$$g(t,x,\cdot) \le^{**} \bar{g}(t,x,\cdot) \quad \text{for a.e. } (t,x) \in (0,T) \times (0,1),$$

and

$$u_0 \le \bar{u}_0 \qquad \text{on } [0,1].$$

Let u and $\bar{u}$ be solutions of CP_0 *and* $\overline{CP}_0$ *on* $[0,T]$ *in* $W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, *respectively. Then,*

$$u \le \bar{u} \text{ on } [0,T] \times [0,1].$$

The above lemmas can be proved in a way similar to that of Kenmochi [3; Lemmas 3.3 and 3.4], so we omit their proofs.

In proving Theorem 1.1, we need another initial-boundary value problem $CP_1(l;\ b_1^t;\ g;\ f_1;\ u_0)$ formulated below:

$$\rho(u)_t - a(u_x)_x + h = f_1 \qquad \text{in } Q_l^-(T),$$

$$h(t,x) \in g(t,x,u(t,x)) \text{ for a.e. } (t,x) \in Q_l^-(T),$$

$$u(t,x) = 0 \qquad \text{for } (t,x) \in \overline{Q_l^+(T)},$$

$$-a(u_x(t,1-)) \in \partial b_1^t(u(t,1)) \qquad \text{for a.e. } t \in [0,T],$$

$$u(0,x) = u_0(x) \qquad \text{for } 0 \le x \le 1,$$

where $x = l(t)$ is a given curve in $C([0,T])$ such that $0 < l(t) < 1$ and u_0 is a given initial datum in X.

<u>LEMMA 3.1'</u>. *Assume (a1) ~ (a5) hold, and* $u_0 \in X$, $u_0(1) \in D(b_1^0)$ *and*

$$u_0 \le 0 \text{ on } [l(0), 1], \quad u_0 = 0 \text{ on } [0, l(0)].$$

Let u be a solution of $CP_1(l;\ b_1^t;\ g;\ f_1;\ u_0)$ in $W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$. Then

$$u \le 0 \text{ on } Q_l^-(T).$$

Concerning problem CP_1, the comparison result similar to Lemma 3.2 holds, too.

4. Energy inequalities

LEMMA 4.1. *Under the same assumptions as in Theorem 1.1, if $\{u, l\}$ is a solution to SP on $[0,T]$ and for a positive constant δ*

$$\delta \le l(t) \le 1 - \delta \text{ on } [0,T],$$

then it holds that for any $0 \le s \le t \le T$

$$\begin{aligned} &\int_s^t |l'(\tau)|^{p'+2}\, d\tau \\ \text{(4.1)}\quad &\le C_\delta [\,|u_x|^p_{L^\infty(s,t;L^p(0,1))} \int_s^t (|\rho(u)_\tau(\tau)|_H^2 + |h(\tau)|_H^2 + |f_0(\tau)|_H^2 + \\ &\quad |f_1(\tau)|_H^2) d\tau + (t-s)|u_x|^{3p-2}_{L^\infty(s,t;L^p(0,1))}\,], \end{aligned}$$

where $1/p + 1/p' = 1$ and C_δ is a positive constant depending only on δ, p and a_1.

Proof. First, we recall the following inequality in Sobolev spaces: For any positive number δ,

$$\text{(4.2)}\qquad |v|_{L^\infty(0,\delta)} \le C(\delta, p)(|v|^{p'/(p'+2)}_{L^{p'}(0,\delta)} |v_x|^{2/(p'+2)}_{L^2(0,\delta)} + |v|_{L^{p'}(0,\delta)}),$$

where $1/p + 1/p' = 1$ and $C(\delta, p)$ is a positive constant depending only on δ and p. We note here that $C(\delta, p)$ is chosen so as to be bounded in R, as long as δ varies in any compact subset of $(0,\infty)$ for $p \ge 2$ (cf. O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva [5; Chap. 2, Theorem 2.2]). By virtue of (0.3) and (4.2), it is easy to check that (4.1) holds. q.e.d.

Proposition 4.1. *Under the same assumptions as in Lemma 4.1, for the solution $\{u, l\}$ to SP on $[0,T]$, it holds that for any $0 \le s \le t \le T$,*

$$\begin{aligned} &\int_0^1 A(u_x(t,x))dx + b_0^t(u(t,0)) + b_1^t(u(t,1)) + \frac{1}{2C_\rho}\int_s^t |\rho(u)_\tau(\tau)|_H^2\, d\tau \\ &\le \int_0^1 A(u_x(s,x))dx + b_0^s(u(s,0)) + b_1^s(u(s,1)) + \int_s^t |\alpha_0'(\tau)| F(\tau, u(\tau))d\tau \\ &+ \int_s^t |\alpha_1'(\tau)|(|a(u_x(\tau,0+))| + |a(u_x(\tau,1-))|)F(\tau,u(\tau))^{1/p} d\tau \\ &+ C_\rho^3 \int_s^t (|h(\tau)|_H^2 + |f_0(\tau)|_H^2 + |f_1(\tau)|_H^2)d\tau \end{aligned}$$

where $A(r) = \int_0^r a(s)ds$,

$$F(t,z) = B_1(b_0^t(z(0)) + B_2|z(0)|^p + B_3) + B_1(b_1^t(z(1)) + B_2|z(1)|^p + B_3)$$

with some positive constants B_1, B_2 and B_3 determined only by T,

$|\alpha_0'|_{L^1(0,T)}$, $|\alpha_1'|_{L^1(0,T)}$ and b_i^0, $i = 0, 1$.

LEMMA 4.2. *Suppose that (a1) ~ (a6) hold, and let $\{u, l\}$ be a solution of SP on $[0,T]$. Then for a sufficiently small δ with $\delta > 0$ and any $0 \le s \le t \le T$,*

$$E_0(t) + \frac{1}{2C_\rho}\int_s^t\int_0^\delta |\rho(u)_\tau(\tau,x)|^2 dxd\tau + \frac{1}{2C_\rho}\int_s^t\int_{1-\delta}^1 |\rho(u)_\tau(\tau,x)|^2 dxd\tau$$
$$\le E_0(s) + \int_s^t|\alpha_0'(\tau)|(|a(u_x(\tau,0+))| + |a(u_x(\tau,1-))|)F_0(\tau)^{1/p}d\tau$$
$$+ \int_s^t|\alpha_1'(\tau)|F_0(\tau)d\tau + \int_s^t[a(u_x(\tau,\delta))u(\tau,\delta) - a(u_x(\tau,1-\delta))u(\tau,1-\delta)]d\tau$$
$$+ C_\rho^3\int_s^t\int_0^\delta(|f_0(\tau,x)|^2 + |h(\tau,x)|^2)dxd\tau + C_\rho^3\int_s^t\int_0^\delta(|f_1(\tau,x)|^2 + |h(\tau,x)|^2)dxd\tau$$

where $E_0(t) = \int_0^\delta A(u_x(t,x))dx + \int_{1-\delta}^1 A(u_x(t,x))dx + b_0^t(u(t,0)) + b_1^t(u(t,1))$,
$F_0(t) = B_1(b_0^t(u(t,0)) + B_2|u(t,0))|^p + B_3) + B_1(b_1^t(u(t,1)) + B_2|u(t,1)|^p + B_3)$.

For precise definitions of B_1, B_2, B_3 and δ, and the proof of Lemma 4.2, see Kenmochi [2; Section 4].

LEMMA 4.3. *Under the same assumptions as in Proposition 4.1, for any $0 \le s \le t \le T$ and for any sufficiently small $\delta > 0$ we have:*

$$\int_\delta^{1-\delta} A(u_x(t,x))dx + \frac{1}{2C_\rho}\int_s^t\int_\delta^{1-\delta}|\rho(u)_\tau|^2 dxd\tau$$
$$\le \int_\delta^{1-\delta} A(u_x(s,x))dx + \int_s^t a(u_x(\tau,1-\delta))u_\tau(\tau,1-\tau)d\tau \tag{4.3}$$
$$- \int_s^t a(u_x(\tau,\delta))u_\tau(\tau,\delta)d\tau + C_\rho^3\int_s^t\int_\delta^{1-\delta}|h(\tau,x)|^2 dxd\tau$$
$$+ C_\rho^3\int_s^t\int_\delta^{l(\tau)}|f_0(\tau,x)|^2 dxd\tau + C_\rho^3\int_s^t\int_{l(\tau)}^{1-\delta}|f_1(\tau,x)|^2 dxd\tau.$$

Proof. We choose some positive number $\mu_0 \in (0,T]$ and put

$$\hat{u}(t,x) = \begin{cases} u(t,x) & \text{for } 0 \le t \le T \text{ and } \delta \le x \le 1-\delta, \\ u_0(x) & \text{for } -\mu_0 \le t < 0 \text{ and } \delta \le x \le 1-\delta. \end{cases}$$

For any $t \in [0,T]$ and $\mu \in (0,\mu_0]$,

$$\int_0^t\int_\delta^{1-\delta}\rho(u)_\tau(\tau,x)\frac{1}{\mu}(\hat{u}(\tau,x) - \hat{u}(\tau-\mu,x))dxd\tau$$
$$= \frac{1}{\mu}\int_0^t\int_\delta^{l(\tau)} a(u_x)_x(\tau,x)(\hat{u}(\tau,x) - \hat{u}(\tau-\mu,x))dxd\tau \tag{4.4}$$
$$+ \frac{1}{\mu}\int_0^t\int_\delta^{l(\tau)} f_0(\tau,x)(\hat{u}(\tau,x) - \hat{u}(\tau-\mu,x))dxd\tau$$
$$+ \frac{1}{\mu}\int_0^t\int_{l(\tau)}^{1-\delta} a(u_x)_x(\tau,x)(\hat{u}(\tau,x) - \hat{u}(\tau-\mu,x))dxd\tau$$
$$+ \frac{1}{\mu}\int_0^t\int_{l(\tau)}^{1-\delta} f_1(\tau,x)(\hat{u}(\tau,x) - \hat{u}(\tau-\mu,x))dxd\tau$$
$$- \frac{1}{\mu}\int_0^t\int_\delta^{1-\delta} h(\tau,x)(\hat{u}(\tau,x) - \hat{u}(\tau-\mu,x))dxd\tau.$$

Now we estimate the first and third terms of the right hand side of (4.4) in the following manner:

$$\frac{1}{\mu}\int_0^t\int_{l(\tau)}^{1-\delta}a(u_x)_x(\tau,x)(\hat{u}(\tau,x)-\hat{u}(\tau-\mu,x))dxd\tau$$
$$+\frac{1}{\mu}\int_0^t\int_{\delta}^{l(\tau)}a(u_x)_x(\tau,x)(\hat{u}(\tau,x)-\hat{u}(\tau-\mu,x))dxd\tau$$
$$=-\frac{1}{\mu}\int_0^t\int_{\delta}^{1-\delta}a(u_x)(\tau,x)(\hat{u}_x(\tau,x)-\hat{u}_x(\tau-\mu,x))dxd\tau$$
$$+\frac{1}{\mu}\int_0^t a(u_x(\tau,l(\tau)-))(\hat{u}(\tau,l(\tau))-\hat{u}(\tau-\mu,l(\tau)))d\tau$$
$$-\frac{1}{\mu}\int_0^t a(u_x(\tau,\delta))(\hat{u}(\tau,\delta)-\hat{u}(\tau-\mu,\delta))d\tau$$
$$+\frac{1}{\mu}\int_0^t a(u_x(\tau,1-\delta))(\hat{u}(\tau,1-\delta)-\hat{u}(\tau-\mu,1-\delta))d\tau$$
$$-\frac{1}{\mu}\int_0^t a(u_x(\tau,l(\tau)+))(\hat{u}(\tau,l(\tau))-\hat{u}(\tau-\mu,l(\tau)))d\tau$$
$$\le\frac{1}{\mu}\int_0^t\int_{\delta}^{1-\delta}(A(\hat{u}_x(\tau-\mu,x))-A(u_x(\tau,x)))dxd\tau$$
$$+\frac{1}{\mu}\int_0^t l'(\tau)\hat{u}(\tau-\mu,l(\tau))d\tau$$
$$+\frac{1}{\mu}\int_0^t a(u_x(\tau,1-\delta))(\hat{u}(\tau,1-\delta)-\hat{u}(\tau-\mu,1-\delta))d\tau$$
$$-\frac{1}{\mu}\int_0^t a(u_x(\tau,\delta))(\hat{u}(\tau,\delta)-\hat{u}(\tau-\mu,\delta))d\tau.$$

Next, we show that the following inequality holds:

$$\liminf_{\mu\downarrow 0}\ -\frac{1}{\mu}\int_0^t l'(\tau)\hat{u}(\tau-\mu,l(\tau))d\tau\ge 0. \tag{4.5}$$

In fact, by (0.3) and Lemma 3.1 for a.e. $\tau\in[0,T]$,

$$\liminf_{\mu\downarrow 0}\ -l'(\tau)\hat{u}(\tau-\mu,l(\tau))\ge 0. \tag{4.6}$$

On the other hand,

$$\frac{1}{\mu}|l'(\tau)\hat{u}(\tau-\mu,l(\tau))|$$
$$=\frac{1}{\mu}|l'(\tau)(\hat{u}(\tau-\mu,\hat{l}(\tau-\mu))-\hat{u}(\tau-\mu,l(\tau)))|$$
$$\le a_o^{1/(1-p)}|l'(\tau)||\frac{1}{\mu}(\hat{l}(\tau-\mu)-\hat{l}(\tau))||a(\hat{u}_x(\tau-\mu))|^{1/(p-1)}_{L^\infty(0,1)}, \tag{4.7}$$

where

$$\hat{l}(t)=\begin{cases} l(t) & \text{for } t\in[0,T],\\ l_o & \text{for } t\in[-\mu_o,0),\end{cases}$$

From Lemma 4.1 we deduce that $l'\in L^{p'+2}(0,T)$ and

$$\frac{1}{\mu}(\hat{l}(\cdot)-\hat{l}(\cdot-\mu))\to l'(\cdot)\qquad \text{in } L^{p'+2}(0,t)\quad\text{as }\mu\downarrow 0. \tag{4.8}$$

The inequality (4.2) shows that

$$|a(u_x(\cdot))|^{1/(p-1)}_{L^\infty(0,1)}\in L^{(p'+2)/p'}(0,T).$$

Accordingly,

$$\int_0^t|l'(\tau)||\frac{1}{\mu}(\hat{l}(\tau-\mu)-\hat{l}(\tau))||a(\hat{u}_x(\tau-\mu))|^{1/(p-1)}_{L^\infty(0,1)}d\tau$$
$$\to\int_0^t|l'(\tau)|^2|a(u_x(\tau))|^{1/(p-1)}_{L^\infty(0,1)}d\tau\qquad\text{as }\mu\downarrow 0.$$

Combining this with (4.6), (4.7) and Fatou's Lemma, we conclude that (4.5) holds. Letting $\mu\downarrow 0$ in (4.4), we see that for a.e. $t\in[0,T]$

$$\frac{1}{C_\rho}\int_0^t\int_\delta^{1-\delta}|\rho(u)_\tau|^2dxd\tau + \int_\delta^{1-\delta}A(u_x(t,x))dx$$

$$\le \int_\delta^{1-\delta}A(u_{0,x}(x))dx + \int_0^t a(u_x(\tau,1-\delta))u_\tau(\tau,1-\delta)d\tau - \int_0^t a(u_x(\tau,\delta))u_\tau(\tau,\delta)d\tau$$
$$- \int_0^t\int_\delta^{1-\delta}h(\tau,x)u_\tau(\tau,x)dxd\tau + \int_0^t\int_\delta^{l(\tau)}f_0(\tau,x)u_\tau(\tau,x)dxd\tau$$
$$+ \int_0^t\int_{l(\tau)}^{1-\delta}f_1(\tau,x)u_\tau(\tau,x)dxd\tau.$$

Hence, for any $t \in [0,T]$,

$$\frac{1}{2C_\rho}\int_0^t\int_\delta^{1-\delta}|\rho(u)_\tau|^2dxd\tau + \int_\delta^{1-\delta}A(u_x(t,x))dx$$

$$\le \int_\delta^{1-\delta}A(u_{0,x}(x))dx + \int_0^t a(u_x(\tau,1-\delta))u_\tau(\tau,1-\delta)d\tau - \int_0^t a(u_x(\tau,\delta))u_\tau(\tau,\delta)d\tau$$
$$+ C_\rho^3\int_0^t\int_\delta^{1-\delta}h(\tau,x)dxd\tau + C_\rho^3\int_0^t\int_\delta^{l(\tau)}|f_0(\tau,x)|^2dxd\tau + C_\rho^3\int_0^t\int_{l(\tau)}^{1-\delta}|f_1(\tau,x)|^2dxd\tau.$$

Thus we have Lemma 4.3 with $s = 0$ and any $t \in [0,T]$. By repeating the same argument as above in the case of initial time $s \in (0,T)$, we get Lemma 4.3.

q.e.d.

By Lemmas 4.2 and 4.3, clearly Proposition 4.1 is obtained.

5. Proof of Theorem 1.1

Suppose $T^* < T_0$ and either (b) or (c) does not holds. Then, there would exists a sequence $\{t_n\}$ with $t_n \uparrow T^*$ (as $n \to \infty$) and a positive number δ_0 such that

$$\delta_0 < l(t_n) < 1 - \delta_0 \qquad \text{for any } n.$$

For $i = 0, 1$, let $v^{(i)}$ be the solution of $(IBP)_i$ on $[0,T^*]$. By the definition of T_0, there is a positive constant M such that for $i = 0, 1$,

$$|v^{(i)}| \le M - 1 \qquad \text{on } [0,T^*] \times [0,1].$$

Further it follows from Lemmas 3.1 and 3.2 that

$$0 \le u \le v^{(0)} \qquad \text{on } Q_l^+(T^*),$$
$$v^{(1)} \le u \le 0 \qquad \text{on } Q_l^-(T^*).$$

Hence

(5.1) $\qquad |u| \le M - 1 \qquad$ on $[0,T^*] \times [0,1]$.

Now, let $\{V_n^{(i)}, L_n^{(i)}\}$ be the solution of $SP_n^{(i)} = SP(\rho;\ a;\ b_0^{(i)},\ b_1^{(i)};\ g_n;\ f_{0,n}^{(i)},\ f_{1,n}^{(i)};\ V_0^{(i)},\ L_0^{(i)})$ for $i = 0, 1$ and $n = 1, 2, \cdots$, where

$$b_0^{(0)}(r)\ (\text{resp. } b_1^{(0)}(r)) = \begin{cases} 0 & \text{for } r = M \ (\text{resp. } r = 0), \\ \infty & \text{otherwise,} \end{cases}$$

$$b_0^{(1)}(r)\ (\text{resp. } b_1^{(1)}(r)) = \begin{cases} 0 & \text{for } r = M \ (\text{resp. } r = -M), \\ \infty & \text{otherwise,} \end{cases}$$

$$g_n(t,x,r) = g(t + t_n, x, r) \qquad \text{for } (t,x) \in R_+ \times (0,1) \text{ and } r \in R,$$

$$f_{o,n}^{(o)}(t,x) = f_o(t + t_n, x) \quad \text{for } (t,x) \in R_+ \times (0,1),$$
$$f_{1,n}^{(o)}(t,x) = 0 \quad \text{for } (t,x) \in R_+ \times (0,1),$$
$$f_{o,n}^{(1)}(t,x) = 0 \quad \text{for } (t,x) \in R_+ \times (0,1),$$
$$f_{1,n}^{(1)}(t,x) = f_1(t + t_n, x) \quad \text{for } (t,x) \in R_+ \times (0,1),$$
$$V_o^{(o)}(x) = \begin{cases} M & \text{for } 0 \le x \le 1 - \delta_o, \\ -\frac{M}{\delta_o}\{x - (1 - \delta_o)\} + M & \text{for } 1 - \delta_o < x \le 1, \end{cases}$$
$$V_o^{(1)}(x) = \begin{cases} -\frac{M}{\delta_o}(x - \delta_o) - M & \text{for } 0 \le x \le \delta_o, \\ -M & \text{for } \delta_o < x \le 1, \end{cases}$$
$$L_o^{(o)} = 1 - \delta_o \text{ and } L_o^{(1)} = \delta_o.$$

By Theorem 2.1, there exists T_1 such that for $i = 0, 1$ and $n = 1, 2, \cdots$, $SP_n^{(i)}$ has a unique solution $\{V_n^{(i)}, L_n^{(i)}\}$ on $[0,T_1]$ and

$$\delta_o/2 \le L_n^{(1)}(t) \le L_n^{(o)}(t) \le 1 - \delta_o/2 \quad \text{for } t \in [0,T_1] \text{ and } n = 1, 2, \cdots.$$

In this case, on account of (5.1), it follows from the usual comparison result for Stefan problems with Dirichlet boundary conditions that

$$L_n^{(1)}(t - t_n) \le l(t) \le L_n^{(o)}(t - t_n) \text{ for any } t \in [t_n, T^*) \text{ with } T^* - t_n \le T_1 \text{ and } n = 1, 2, \cdots.$$

Hence

$$(5.2) \qquad 0 < \inf\{l(t);\ 0 \le t < T^*\} \le \sup\{l(t);\ 0 \le t < T^*\} < 1.$$

From (5.2), Lemma 3.1 and Proposition 4.1 we obtain that $u(T^*) \in X$, $u(T^*, i) \in D(b_i^{T^*})$ $(i = 0, 1)$, $u(T^*, \cdot) \ge 0$ on $[0, l(T^*)]$ and $u(T^*, \cdot) \le 0$ on $[l(T^*), 1]$, so that on account of Theorem 2.1, $SP(\rho;\ a;\ b_o^t,\ b_1^t;\ g;\ f_o,\ f_1;\ u(T^*),\ l(T^*))$ has a solution on a certain interval $[T^*, T']$, $T' > T^*$. Therefore SP has a solution on $[0,T']$. This contradicts the definition of T^*. Thus the case (b) or (c) holds true, provided $T^* < T_o$. q.e.d.

6. Weak compatibility condition for the Stefan data

The purpose of this section is to establish existence and uniqueness theorems for Stefan problems as well as the behavior for solutions under weak compatibility conditions for the Stefan data.

In this section, we denote by $SP^* = SP^*(\rho;\ a;\ b_o^t,\ b_1^t;\ g;\ f_o,\ f_1;\ u_o,\ l_o)$ on $[0,T]$, $0 < T < \infty$, the Stefan problem with (0.3) and (0.5) replaced by the following (6.1) and (6.2), respectively:

$$(6.1) \qquad \begin{cases} u(t, l(t)) = 0 & \text{for } t \in (0,T], \\ l'(t) = -a(u_x(t, l(t)-)) + a(u_x(t, l(t)+)) & \text{for a.e. } t \in [0,T], \\ l(0) = l_o; \end{cases}$$

$$(6.2) \qquad u(0,x) = u_o(x) \quad \text{for a.e. } x \in [0,1].$$

We say that $\{u,\ l\}$ is a solution of SP^* on $[0,T]$, if $u \in C([0,T];H) \cap W^{1,2}_{loc}((0,T];H) \cap L^p(0,T;X) \cap L^\infty_{loc}((0,T];X)$, $b_i^{(\cdot)}(u(\cdot,i)) \in L^1(0,T) \cap L^\infty_{loc}((0,T])$, $i = 0,\ 1$, $l \in C([0,T]) \cap W^{1,2}_{loc}((0,T])$, $0 < l < 1$ on $[0,T]$, and (0.1), (0.2), (6.1), (0.4) and (6.2) are satisfied. Also, we say that for $0 < T' \le \infty$, $\{u,\ l\}$ is a solution of SP^* on $[0,T')$, if it is a solution of SP^* on $[0,T]$ for every $0 < T < T'$ in the above sence.

We suppose that the weak compatibility conditions on the data b_0^t, b_1^t, l_0 and u_0 consist of the following (a5)' and (a6)':

(a5)' In addition to (a5), suppose that there exist proper l.s.c. convex functions $\bar{b}_0$ and $\bar{b}_1$ such that

$$b_0^t \le^* \bar{b}_0 \ \text{ on } R \text{ and } \bar{b}_1 \le^* b_1^t \text{ on } R \text{ for } t \ge 0,$$

(a6)' $0 < l_0 < 1$, $u_0 \in L^\infty(0,1)$ and there exists a sequence $\{u_{0,n}\} \subset X$ having the following properties (i) ~ (iii):

(i) For $i = 0,\ 1$ and each n, $u_{0,n}(i) \in D(b_i^0)$, $u_{0,n} \ge 0$ on $[0,l_0]$ and $u_{0,n} \le 0$ on $[l_0,1]$;

(ii) there exist functions $v_0^{(o)}$ and $v_0^{(1)}$ in X such that

$$v_0^{(o)}(0) \in D(\bar{b}_0),\ v_0^{(1)} \in D(\bar{b}_1),\ v_0^{(o)}(1) = v_0^{(1)}(0) = 0, \text{ and}$$

$$v_0^{(1)} \le u_{0,n} \le v_0^{(o)} \ \text{ on } [0,1];$$

(iii) $u_{0,n} \to u_0$ in H.

<u>REMARK 6.1</u>. *It is easily to see that inequality (2.1) holds true under the weak compatibility conditions, so that the solution of SP^* is unique.*

The next theorem is concerned with the existence of a solution to SP^*.

<u>THEOREM 6.1</u>. *We suppose that the assumptions (a1) ~ (a4), (a5)' and (a6)' hold. Then, for some positive number T', SP^* has a solution on $[0,T']$ such that*

$$u \in L^\infty((0,T') \times (0,1)),$$

$$t^{1/2}u_t \in L^2(0,T';H),$$

$$t^{1/p}u_x \in L^\infty(0,T';L^p(0,1)),$$

and

$$t^{2/(p'+2)}l' \in L^{p'+2}(0,T'),$$

where $1/p + 1/p' = 1$.

In order to prove Theorem 6.1 we need one-phase Stefan problems. We denote by $SP_0^*(\rho;\ a;\ b_0^t;\ g;\ f_0;\ u_0,\ l_0)$ on $[0,T]$ the problem of finding $u = u(t,x)$ on $Q(T)$ and $x = l(t)$, $0 < l(t) < 1$ on $[0,T]$ such that

(6.3) $$\rho(u)_t - a(u_x)_x + h = f_0 \quad \text{in } Q_l^+(T),$$

(6.4) $h(t,x) \in g(t,x,u(t,x))$ for a.e. $(t,x) \in Q_l^+(T)$,

(6.5) $u(0,x) = u_o(x)$ for a.e. $x \in [0,1]$,

(6.6) $a(u_x(t,0+)) \in \partial b_o^t(u(t,0))$ for a.e. $t \in [0,T]$,

(6.7) $u(t,x) = 0$ for any $t \in (0,T]$ and $l(t) \le x \le 1$,

(6.8) $$\begin{cases} l'(t) = -a(u_x(t,l(t)-)) \text{ for a.e. } t \in [0,T], \\ l(0) = l_o. \end{cases}$$

A pair $\{u, l\}$ is called a solution of SP_o^* on $[0,T]$, if $u \in C([0,T];H) \cap W_{loc}^{1,2}((0,T];H) \cap L^p(0,T;X) \cap L_{loc}^\infty((0,T];X)$, $l \in C([0,T]) \cap W_{loc}^{1,2}((0,T])$, $b_o^{(\cdot)}(u(\cdot,0)) \in L^1(0,T) \cap L_{loc}^\infty((0,T])$ and (6.3) ~ (6.8) hold.

PROPOSITION 6.1. *Suppose that (a1) ~ (a4) and (a5)′ hold. Let* $0 < l_o < 1$ *and* $u_o \in L^\infty(0,1)$ *such that there exists a sequence* $\{u_{o,n}\} \subset X$ *satisfying the following conditions (i) ~ (iii):*

(i) For each n, $u_{o,n}(0) \in D(b_o^o)$, $u_{o,n} \ge 0$ *on* $[0,l_o]$ *and* $u_{o,n} = 0$ *on* $[l_o,1]$;

(ii) there exists a function $v_o \in X$ *such that* $v_o(0) \in D(\bar{b}_o)$, $v_o(1) = 0$ *and* $0 \le u_{o,n} \le v_o$ *on* $[0,1]$;

(iii) $u_{o,n} \to u_o$ *in H.*

Then for a certain positive number T_1 SP_o^* *has one and only one solution* $\{u, l\}$ *on* $[0,T_1]$ *such that* l *is non-decreasing on* $[0,T_1]$ *and*

$$t^{2/(p'+2)} l' \in L^{p'+2}(0,T_1),$$
$$u \in L^\infty((0,T_1) \times (0,1)),$$
$$t^{1/2} u_t \in L^2(0,T_1;H),$$

and

$$t^{1/p} u_x \in L^\infty(0,T_1;L^p(0,1)).$$

The next proposition is concerned with convergence of solution to SP_o^*.

PROPOSITION 6.2. *Let* ρ, a, b_o^t, g, f_o, u_o *and* l_o *be as in Proposition 6.1. Also, let* $0 < l_{o,n} < 1$ *and* $u_{o,n} \in H$ *such that for each* $n = 1, 2, \cdots$ *there exists a sequence* $\{u_{o,n}^{(m)}\} \subset X$ *satisfying the following conditions (i) ~ (iii):*

(i) For each m, $u_{o,n}^{(m)}(0) \in D(b_o^o)$, $u_{o,n}^{(m)} \ge 0$ *on* $[0,l_{o,n}]$ *and* $u_{o,n}^{(m)} = 0$ *on* $[l_{o,n},1]$;

(ii) there exists a function $v_o \in X$ *such that* $v_o(0) \in D(\bar{b}_o)$, $v_o(1) = 0$ *and* $0 \le u_{o,n}^{(m)} \le v_o$ *on* $[0,1]$ *for each m;*

(iii) $u_{o,n}^{(m)} \to u_{o,n}$ *in H as* $m \to \infty$.

Further suppose that

$$l_{o,n} \to l_o \text{ and } u_{o,n} \to u_o \text{ in } H \text{ as } n \to \infty,$$

and that $SP_0^* = SP_0^*(\rho;\ a;\ b_0^t;\ g;\ f_0;\ u_0,\ l_0)$ *has a solution* $\{u,\ l\}$ *on an interval* $[0,T]$, $0 < T < \infty$. *Then,* $(SP_0^*)_n = SP_0^*(\rho;\ a;\ b_0^t;\ g;\ f_0;\ u_{0,n},\ l_0)$ *has a solution* $\{u_n,\ l_n\}$ *on the same interval* $[0,T]$. *Moreover,*

$$u_n \to u \text{ in } C([0,T];H) \text{ and } L^p(0,T;X),$$

and

$$l_n \to l \text{ in } C([0,T]).$$

The above Propositions 6.1 and 6.2 can be proved in a way similar to that of Kenmochi [3; Propositions 6.1 and 6.2].

Also, we consider another one-phase Stefan problem $SP_1^* = SP_1^*(\rho;\ a;\ b_1^t;\ g;\ f_1;\ u_0,\ l_0)$ on $[0,T]$ which is the problem of finding $u = u(t,x)$ on $Q(T)$ and $x = l(t)$, $0 < l < 1$ on $[0,T]$ such that

(6.9) $\quad \rho(u)_t - a(u_x)_x + h = f_1 \quad$ in $Q_l^-(T)$,

(6.10) $\quad h(t,x) \in g(t,x,u(t,x)) \quad$ for a.e. $(t,x) \in Q_l^-(T)$,

(6.11) $\quad u(0,x) = u_0(x) \quad$ for a.e. $x \in [0,1]$,

(6.12) $\quad -a(u_x(t,1-)) \in \partial b_1^t(u(t,1)) \quad$ for a.e. $t \in [0,T]$,

(6.13) $\quad u(t,x) = 0 \quad$ for any $t \in (0,T]$ and $0 \le x \le l(t)$,

(6.14) $\quad \begin{cases} l'(t) = a(u_x(t,l(t)+)) & \text{for a.e. } t \in [0,T], \\ l(0) = l_0. \end{cases}$

We say that $\{u,\ l\}$ is a solution of SP_1^* on $[0,T]$, if $u \in C([0,T];H) \cap W^{1,2}_{loc}((0,T];H) \cap L^p(0,T;X) \cap L^\infty_{loc}((0,T];X)$, $b_1^{(\cdot)}(u(\cdot,1)) \in L^1(0,T) \cap L^\infty_{loc}((0,T])$, $l \in C([0,T]) \cap W^{1,2}_{loc}((0,T])$ and (6.9) ~ (6.14) are satisfied.

Concerning problem SP_1^*, the existence reslt similar to Proposition 6.1 and the convergence result similar to Proposition 6.2 hold.

<u>Proof of Theorem 6.1</u>. Let $\{u_{0,n}\} \subset X$ be a sequence which satisfies properties in (a6)'. Put $(SP^*)_n = SP^*(\rho;\ a;\ b_0^t,\ b_1^t;\ g;\ f_0,\ f_1;\ u_{0,n},\ l_0)$, and let $\{u_n,\ l_n\}$ be the solution of $(SP^*)_n$ on $[0,T_n)$, where $[0,T_n)$ is the maximal interval of existence of the solution. For $i = 0,\ 1$, let $v^{(i)}$ be the solution of $(IBP)_i$ with $v^{(i)}(0,x) = v_0^{(i)}(x)$ on $[0,\hat{T}]$, where $\hat{T}$ is a positive number. Then we can choose a positive number M such that $|v^{(i)}| \le M$ on $[0,\hat{T}] \times [0,1]$, for $i = 0,\ 1$, in fact, $v^{(i)} \in W^{1,2}(0,\hat{T};H) \cap L^\infty(0,\hat{T};X)$. By Lemmas 3.1 and 3.2,

$$0 \le u_n \le v^{(0)} \text{ on } Q_l^+(T_n^*), \quad v^{(1)} \le u_n \le 0 \text{ on } Q_l^-(T_n^*) \text{ for } n = 1,\ 2,\ \cdots,$$

where $T_n^* = \min\{T_n,\ \hat{T}\}$. Now, choose a constant $M > 0$ so that

(6.15) $\quad |u_n| \le M \quad$ on $[0,T_n^*) \times [0,1]$ for $n = 1,\ 2,\ \cdots$.

Also, put

$$z_{0,n} = u_{0,n}^+,\ z_0 = u_0^+,$$

and

$$z_{1,n} = -u^-_{o,n},\ z_1 = -u^-_o,$$

and denote by $\{u^i_n, l^i_n\}$, $i = 0, 1$, the solution of one-phase Stefan problem $SP^*_i(\rho;\ a;\ b^t_i;\ g;\ f_i;\ z_{i,n},\ l_o)$ on $[0,\tilde{T}]$, for some $\tilde{T} > 0$ independent of n; by Proposition 6.2, such a $\tilde{T}$ certainly exists. Moreover, an extensive use of Theorem 2.2 implies that

(6.16) $\quad u^1_n \le u_n \le u^0_n$ on $[0,\tilde{T}_n) \times [0,1]$, $l^1_n \le l_n \le l^0_n$ on $[0,\tilde{T}_n)$,

where $\tilde{T}_n = min\ \{T^*_n, \tilde{T}\}$. It follows from Proposition 6.2 again that

$$(6.17)\quad \left[\begin{array}{ll} l^i_n \to l^i & \text{in } C([0,\tilde{T}]),\ i = 0, 1, \\ u^i_n \to u^i & \text{in } C([0,\tilde{T}];H),\ i = 0, 1, \end{array}\right.$$

where $\{u^i, l^i\}$ is the solution of $SP^*_i(\rho;\ a;\ b^t_i;\ g;\ f_i;\ z_i,\ l_o)$ on $[0,\tilde{T}]$, $i = 0, 1$. We note here that there are positive constants δ, T_o such that

$$\delta \le l^i_n \le 1 - \delta \text{ on } [0,T_o] \text{ for } i = 0,\ 1 \text{ and large } n,$$

and by (6.16)

$$\delta \le l_n(t) \le 1 - \delta \text{ for } t \in [0,\tilde{T}_n) \cap [0,T_o] \text{ and large } n.$$

Hence, just as in the proof of Theorem 1.1, we see that $\tilde{T}_n > T_o$ for large n, and by virtue of Proposition 4.1 and Lemma 4.1 $\{u_n\}$ is bounded in $W^{1,2}(T_o - \varepsilon, T_o;H) \cap L^\infty(T_o - \varepsilon, T_o;X)$, $\{l_n\}$ is bounded in $W^{1,p'+2}(T_o - \varepsilon, T_o)$ and $\{b^{(\cdot)}_i(u_n(\cdot,i))\}$, $i = 0, 1$, is bounded in $L^\infty(T_o - \varepsilon, T_o)$ for every $0 < \varepsilon < T_o$. Using these facts together with (6.16) and (6.17), we can extract a subsequence of $\{n\}$, denoted again by $\{n\}$, such that $u_n \to u$ in $C([0,T];H)$, weakly in $W^{1,2}_{loc}((0,T_o];H)$ and weakly* in $L^\infty_{loc}((0,T_o])$. Besides, it is not difficult to see that limit $\{u,\ l\}$ is the solution of $SP^*(\rho;\ a;\ b^t_o,\ b^t_1;\ f_o,\ f_1;\ u_o,\ l_o)$ on $[0,T_o]$ having the required properties in the statement of Theorem 6.1. q.e.d.

In the following theorem we give a result on the behaivior of the solution to SP^* with weak compatibility conditions for the Stefan data.

THEOREM 6.2. *Under the same assumptions as in Theorem 6.1, one and only one the cases (a), (b), (c) in the statement of Theorem 1.1 always happens.*

This Theorem is a direct consequence of Theorems 1.1 and 6.1.

REFERENCES.

[1], T. Aiki, The existence of solutions to two-phase Stefan problems for nonlinear parabolic equations, to appear in Control Cyb.

[2], P. Knabner, Global existence in a general Stefan-like problem, J. Math. Anal. Appl. 115(1986), 543-559.

[3], N. Kenmochi, Global existence of solutions of two-phase Stefan problems with nonlinear flux conditions described by time-dependent subdifferentials, to appear in Control Cyb.

[4], N. Kenmochi, A new proof of the uniqueness of solutions to two-phase Stefan problems for nonlinear parabolic equations, Tech. Rep. Math. Sci., No. 9., 1987.

[5], O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva, Linear and Quasi-Linear Equations of Parabolic Type, Transl. Math. Monographs 23, Amer. Math. Soc., Providence R. I., 1968.